SKEW YOUNG DIAGRAM METHOD IN SPECTRAL DECOMPOSITION OF INTEGRABLE LATTICE MODELS II: HIGHER LEVELS

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Abstract. The spectral decomposition of the path space of the vertex model associated to the level l representation of the quantized affine algebra $U_q(\widehat{sl}_n)$ is studied. The spectrum and its degeneracy are parametrized by skew Young diagrams and what we call nonmovable tableaux on them, respectively. As a result we obtain the characters for the degeneracy of the spectrum in terms of an alternating sum of skew Schur functions. Also studied are new combinatorial descriptions (spectral decomposition) of the Kostka numbers and the Kostka–Foulkes polynomials. As an application we give a new proof of Nakayashiki–Yamada's theorem about the branching functions of the level l basic representation $l\Lambda_k$ of $\widehat{sl}_n$ and a generalization of the theorem.

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1. Introduction

Study of integrable lattice models often provides new and remarkable expressions of characters for various underlying Lie algebras. For example, the trace of the corner transfer matrix [3] of the $U_q(\widehat{sl}_n)$-*vertex model* leads us to the formula [4]

$$\mathrm{ch}\,\mathcal{L}(\Lambda_k)(q,x) = \sum_{\vec{s}\in\mathcal{S}_k} q^{E(\vec{s})} e^{\mathrm{wt}(\vec{s})}, \qquad k=0,1,\dots,n. \tag{1.1}$$

Here the left hand side is the character of the level-1 integrable module with highest weight Λ_k of untwisted affine Lie algebra $\widehat{sl}_n$. In the right hand side the summation is taken over the *spin configurations* (=*path*) $\vec{s}$; $E(\vec{s})$ and $\mathrm{wt}(\vec{s})$ are the energy and the sl_n-weight of $\vec{s}$.

There is a natural surjection (the *local energy map*) $\rho : \mathcal{S}_k \to \mathrm{Sp}_k$, where Sp_k denotes the *spectrum* of $\mathcal{S}_k$. The map ρ induces the *spectral decomposition*

$$\mathrm{ch}\,\mathcal{L}(\Lambda_k)(q,x) = \sum_{\vec{h}\in\mathrm{Sp}_k} q^{E(\vec{h})}\chi_{\vec{h}}(x), \qquad \chi_{\vec{h}}(x) := \sum_{\vec{s}\in\rho^{-1}(\vec{h})} e^{\mathrm{wt}(\vec{s})}. \tag{1.2}$$

This decomposition is remarkable in the following sense [2, 13]:

(i) The spectrum Sp_k is parametrized by the border strips, which are a class of skew Young diagrams.
(ii) Each subcharacter $\chi_{\vec{h}}$ coincides with the skew Schur function associated to the corresponding border strip.
(iii) Moreover, $\chi_{\vec{h}}$'s are the sl_n-characters of irreducible $Y(sl_n)$-modules, where $Y(sl_n)$ is the Yangian algebra of type sl_n.

This result admits a simple representation-theoretical/physical picture: The integrals of motions of the hamiltonian form a commutative algebra $\mathcal{A}$, and the algebras $\mathcal{A}$ and $Y(sl_n)$ act on $\mathcal{L}(\Lambda_k)$ in a way that they are mutually commutant. The formula (1.2) represents the reciprocal decomposition of the module $\mathcal{L}(\Lambda_k)$ with respect to the actions of $\mathcal{A}$ and $Y(sl_n)$ (cf. [19]).

The paper consists of two main parts.

In the first part (Sections 2 and 3) we attempt to generalize the above properties (i)–(iii) to the levels greater than 1 by considering the *fusion vertex models.* The result is summarized as follows: (i) The spectrum is, again, parametrized by a certain class of skew Young diagrams, and a formula analogous to (1.2) is obtained (Theorem 3.7). (ii) Each subcharacter $\chi_{\vec{h}}$, also denoted by $t_{\kappa(\vec{h})}$, is expressed as an alternating sum of skew Schur functions (Proposition 3.8). (iii) In the case of either $\widehat{sl}_2$ at higher levels or $\widehat{sl}_n$ at level 1, those alternating sums can be reexpressed as a single skew Schur function, thus reproduces the known results in [2, 13]. In the case of $\widehat{sl}_n$ ($n \geq 3$) at levels greater than 1, however, they cannot be, in general, identified with the known characters of irreducible $Y(sl_n)$-modules, i.e., the characters of the tame

modules. Thus, the representation-theoretical interpretation of the decomposition is still missing.

In the second part (Sections 4 and 5) we turn into the study of the *Kostka numbers* $K_{\lambda,\mu}$ and the *Kostka–Foulkes polynomials* $K_{\lambda,\mu}(q)$,

$$K_{\lambda,\mu} = |\mathrm{SST}(\lambda,\mu)|, \qquad K_{\lambda,\mu}(q) = \sum_{T\in \mathrm{SST}(\lambda,\mu)} q^{c(T)},$$

where $\mathrm{SST}(\lambda,\mu)$ is the set of the semistandard tableaux T of shape λ and content μ, and $c(T)$ is the charge by Lascoux and Schützenberger. The set $\mathrm{SST}(\lambda,\mu)$ has a natural decomposition by the *exponents* of a tableau,

$$\mathrm{SST}(\lambda,\mu) = \bigsqcup_{d:\text{exponents}} \mathrm{SST}_d(\lambda,\mu).$$

A key observation is the existence of a bijection $\theta_d : \mathrm{SST}_d(\lambda,\mu) \leftrightarrow \mathrm{LR}_0(\mathrm{Sh}_d(\mu),\lambda)$ (Theorem 4.9), where $\mathrm{LR}_0(\mathrm{Sh}_d(\mu),\lambda)$ is another set of the semistandard tableaux satisfying a certain condition. Thanks to the bijection θ_d, we obtain new combinatorial descriptions (spectral decomposition) of $K_{\lambda,\mu}$ (Corollary 4.10) and $K_{\lambda,(l^m)}(q)$ (Theorem 5.4), which are the main result in the second part.

In Section 6 we identify the Kostka–Foulkes (and other) polynomials with the branching functions of the *truncated characters* of the fusion vertex model. It follows that the decomposition in Theorem 5.4 can be seen as the one induced from that in Theorem 3.7. As an application we obtain an expression of the branching functions of the integrable $\widehat{sl}_n$-modules as limits of the Kostka–Foulkes and other polynomials (Corollaries 6.4 and 6.7).

2. Date-Jimbo-Kuniba-Miwa-Okado correspondence

In this section we review the correspondence between the corner transfer matrix (CTM) spectra of the vertex model of the symmetric representation of $U_q(\widehat{sl}_n)$ and the affine Lie algebra characters of $\widehat{sl}_n$ [4].

2.1. **Energy function.** Let $\overline{\Lambda}_1,\dots,\overline{\Lambda}_{n-1}$ be the fundamental weights of the Lie algebra sl_n, and let $\epsilon_i = \overline{\Lambda}_i - \overline{\Lambda}_{i-1}$ for $i=1,\dots,n$ with $\overline{\Lambda}_0 = \overline{\Lambda}_n = 0$. Then $\{\epsilon_1,\dots,\epsilon_n\}$ is the set of the weights of the irreducible representation of sl_n whose highest weight is $\overline{\Lambda}_1$ (the vector representation). Let

$$B_l = \{ v_{a_1\dots a_l} \mid 1 \le a_1 \le a_2 \le \cdots \le a_l \le n \}.$$

be a basis of the irreducible representation of sl_n whose highest weight is $l\overline{\Lambda}_1$ (the l-fold symmetric tensor representation) such that $\mathrm{wt}(v_{a_1\dots a_l}) = \sum_{i=1}^{l} \epsilon_{a_i}$.

We define the *energy function* $H_l : B_l \times B_l \to \{0,1,\dots,l\}$ as

$$H_l(v_{a_1\dots a_l}, v_{b_1\dots b_l}) = \min_{\sigma:\text{permutation}} \sum_{i=1}^{l} H_1(v_{a_i}, v_{b_{\sigma(i)}}), \tag{2.1}$$

where

$$H_1(v_a, v_b) = \begin{cases} 0 & \text{if } a < b, \\ 1 & \text{if } a \geq b. \end{cases} \tag{2.2}$$

The function H_l is the logarithm of the R-matrix associated to the l-fold symmetric tensor representation of $U_q(\widehat{sl}_n)$ in the limit $q \to 0$ [5].

2.2. **Date-Jimbo-Kuniba-Miwa-Okado (DJKMO) correspondence.** For given two infinite sequences, $\vec{a} = (a_1, a_2, \dots)$ and $\vec{b} = (b_1, b_2, \dots)$, of any kind of objects a_i, b_i, we write $\vec{a} \approx \vec{b}$ if $a_i \neq b_i$ only for finitely many i. We often use a shorthand notation $\vec{a} = (a_1, \dots, a_k, (a_{k+1}, \dots, a_{k+m})^\infty)$ for such a periodic sequence as $\vec{a} = (a_1, \dots, a_k, a_{k+1}, \dots, a_{k+m}, a_{k+1}, \dots, a_{k+m}, \dots)$.

Let $\mathcal{K}_l = \{K = ((k_1, \dots, k_n)^\infty) | k_i \in \mathbf{Z}_{\geq 0}, \sum_{i=1}^n k_i = l.\ \}$. The set $\mathcal{K}_l$ is identified with the set of the dominant integral weights of level l of the untwisted affine Lie algebra $\widehat{sl}_n$ with the correspondence

$$K = (k_i) \leftrightarrow \Lambda(K) := k_n \Lambda_0 + \sum_{i=1}^{n-1} k_i \Lambda_i.$$

For $K = (k_i) \in \mathcal{K}_l$, we define $\vec{s}^{(K)} = (s_i^{(K)})_{i=1}^\infty \in B_l^{\mathbf{N}}$, where

$$s_i^{(K)} := v_{\underbrace{1\dots1}_{k_i}\underbrace{2\dots2}_{k_{i+1}}\dots\underbrace{n\dots n}_{k_{i+n-1}}}, \quad \mathrm{wt}(s_i^{(K)}) = \sum_{j=1}^{n} k_{i+j-1}\epsilon_j.$$

Then an infinite sequence $\vec{s} = (s_i)_{i=1}^\infty \in B_l^{\mathbf{N}}$ is called a *spin configuration* or *path* if $\vec{s} \approx \vec{s}^{(K)}$ for a certain $K \in \mathcal{K}_l$. The set of the spin configurations $\mathcal{S}$, thus, has a natural decomposition

$$\mathcal{S} = \bigsqcup_{K \in \mathcal{K}_l} \mathcal{S}_K, \quad \mathcal{S}_K := \{\vec{s} \mid \vec{s} \approx \vec{s}^{(K)}\}. \tag{2.3}$$

For $\vec{s} = (s_i) \in \mathcal{S}_K$ we define its *energy* $E(\vec{s})$ and sl_n-*weight* $\mathrm{wt}(\vec{s})$ as

$$E(\vec{s}) = \sum_{i=1}^\infty i \left\{ H_l(s_{i+1}, s_i) - H_l(s_{i+1}^{(K)}, s_i^{(K)}) \right\},$$

$$\mathrm{wt}(\vec{s}) = \sum_{i=1}^{n-1} k_i \overline{\Lambda}_i + \sum_{i=1}^\infty \left(\mathrm{wt}(s_i) - \mathrm{wt}(s_i^{(K)}) \right), \quad K = (k_i).$$

As is standard in the character theory of sl_n, we regard $e^{\mathrm{wt}(\vec{s})}$ as a power of the variables $x_1 = e^{\epsilon_1}, x_2 = e^{\epsilon_2}, \dots, x_n = e^{\epsilon_n}$ with the relation $x_1 x_2 \cdots x_n = 1$.

There is a remarkable connection between the partition function of $\mathcal{S}_K$ and an affine Lie algebra character.

Theorem 2.1 (DJKMO correspondence [4, 8])**.** *For a given $K \in \mathcal{K}_l$, let $\mathcal{L}(\Lambda(K))$ be the integrable module of $\widehat{sl}_n$ with the highest weight $\Lambda(K)$. Then the following equality holds:*

$$\mathrm{ch}\,\mathcal{L}(\Lambda(K))(q,x) = \sum_{\vec{s}\in\mathcal{S}_K} q^{E(\vec{s})} e^{\mathrm{wt}(\vec{s})}, \tag{2.4}$$

where $\mathrm{ch}\,\mathcal{L}(\Lambda(K))$ *is the (unnormalized) character of* $\mathcal{L}(\Lambda(K))$ [7]*.*

2.3. **Energy functions and nonmovable tableaux.** Let us describe the energy function H_l in terms of nonmovable tableaux. For this aim, it is convenient to identify the set B_l with the crystal of the l-fold symmetric tensor representation of $U_q(sl(n))$. The latter consists of semistandard tableaux of shape (l) with entries from the set $\{1, 2, \ldots, n\}$. We identify $v_{a_1\ldots a_l} \in B_l$ with a semistandard tableau as

$$v_{a_1\ldots a_l} = \boxed{a_1}\boxed{a_2}\boxed{\quad\cdots\quad}\boxed{a_l}\,.$$

We shall construct the function $H_{l_1,l_2} : B_{l_1} \otimes B_{l_2} \to \mathbf{Z}$ such that H_l in (2.1) is realized as $H_l = H_{l,l}$ under the above identification.

Next we construct the maps (cf. [8, 9])

$$\widetilde{f}_i : B_l \to B_l \cup \{0\}, \quad \widetilde{e}_i : B_l \to B_l \cup \{0\}, \quad 1 \le i \le n-1.$$

Let $b \in B_l$ be a semistandard tableau and $i \in \{1, \ldots, n-1\}$, we define $\varphi_i(b) =$ number of i in b. For each i, $1 \le i \le n-1$, we define a map

$$\widetilde{f}_i : B_l \to B_l \cup \{0\}$$

by the following rule: Let $b' \in B_l$, then

- $\widetilde{f}_i b' = 0$, if $\varphi_i(b') = 0$;
- $\widetilde{f}_i b' = b$ if $\varphi_i(b') > 0$,

where b is obtained from b' by replacing the rightmost i in b' with $i+1$.

Similarly, we define a map

$$\widetilde{e}_i : B_l \to B_l \cup \{0\}$$

by the rule: Let $b \in B_l$, then

- $\widetilde{e}_i b = 0$, if $\varphi_{i+1}(b) = 0$;
- $\widetilde{e}_i b = b'$, if $\varphi_{i+1}(b) > 0$,

where b' is obtained from b by replacing the leftmost $i+1$ in b with i.

We also will use the same symbol $\widetilde{f}_i$ and $\widetilde{e}_i$, $i \in \{1, \ldots, n-1\}$, to denote the maps $\widetilde{f}_i : B_{l_1} \otimes B_{l_2} \to (B_{l_1} \otimes B_{l_2}) \cup \{0\}$ and $\widetilde{e}_i : B_{l_1} \otimes B_{l_2} \to (B_{l_1} \otimes B_{l_2}) \cup \{0\}$ which are defined by the rules

$$\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} (\widetilde{f}_i b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varphi_{i+1}(b_2) \\ b_1 \otimes (\widetilde{f}_i b_2) & \text{otherwise.} \end{cases} \tag{2.5}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} (\tilde{e}_i b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varphi_{i+1}(b_2) \\ b_1 \otimes (\tilde{e}_i b_2) & \text{otherwise.} \end{cases} \tag{2.6}$$

On the right hand sides of (2.5) and (2.6) we assume $0 \otimes B_{l_2} = B_{l_1} \otimes 0 = 0$. It is known [8] that the tensor product of two crystals also becomes a crystal under the above rules.

For each d, $0 \leq d \leq \min(l_1, l_2)$, we define $C_d \subset B_{l_1} \otimes B_{l_2}$ to be the connected component of the following (highest weight) element under the actions of $\tilde{f}_i$, $1 \leq i \leq n-1$,

$$\overbrace{\boxed{1}\ \cdots\ \boxed{1}}^{l_1} \otimes \overbrace{\boxed{1}\ \cdots\ \boxed{1}}^{d+l_2-l_0}\overbrace{\boxed{2}\cdot\cdot\boxed{2}}^{l_0-d},$$

where $l_0 = \min(l_1, l_2)$. As a set, $B_{l_1} \otimes B_{l_2} = \bigsqcup_{d=0}^{\min(l_1,l_2)} C_d$. Moreover as an $U_q(sl(n))$ crystal one has the isomorphism

$$C_d \simeq B\ \overbrace{\underbrace{\boxed{}}_{\min(l_1,l_2)-d}\boxed{}}^{\max(l_1,l_2)+d}, \tag{2.7}$$

where in the RHS of (2.7) the set $B_{(\max(l_1,l_2)+d, \min(l_1,l_2)-d)}$ is the $U_q(sl(n))$ crystal described in [9] via the semistandard tableaux with the specified shape. We define the energy function $H = H_{l_1,l_2} : B_{l_1} \otimes B_{l_2} \to \mathbf{Z}$ by $H(b_1 \otimes b_2) = d$ for any $b_1 \otimes b_2 \in C_d$. Up to an overall additive constant, this agree with the definition from the combinatorial R-matrix in [18]. When $l_1 = l_2$, this definition of the energy function coincides with that in (2.1) as follows:

$$H_{l,l}(v_{a_1 \ldots a_l} \otimes v_{b_1 \ldots b_l}) = H_l(v_{a_1 \ldots a_l}, v_{b_1 \ldots b_l}).$$

Now we are ready to give a description of the energy function $H := H_{l_1,l_2}$ in terms of nonmovable tableaux.

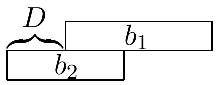
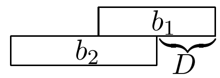
Definition 2.2. A skew shape tableau T is called to be *nonmovable*, if the following conditions are satisfied:

- T is semistandard.
- For each i, let us denote by T_i the tableau which is obtained from T by moving the ith row of T from the right to the left by one box without changing the positions of all other rows. Then the tableau T_i is either of non-skew shape or non-semistandard.

We denote by $\mathrm{NMT}(\lambda)$ the set of all the nonmovable tableaux of shape λ with entries not exceeding n.

To continue, let us introduce one more function on the same space $D = D_{l_1,l_2} : B_{l_1} \otimes B_{l_2} \to \mathbf{Z}$. Let $b_1 \in B_{l_1}$ and $b_2 \in B_{l_2}$ be semistandard tableaux. Then $D =$

$D(b_1 \otimes b_2)$ is uniquely determined by the condition that the following tableau is nonmovable:

 for $l_1 \geq l_2$,  for $l_1 \leq l_2$.

It is clear that $0 \leq D_{l_1,l_2} \leq \min(l_1, l_2)$.

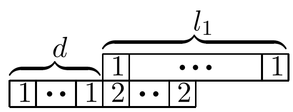
Proposition 2.3.

$$H_{l_1,l_2} = D_{l_1,l_2}. \tag{2.8}$$

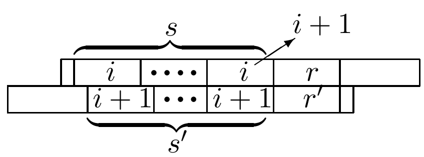
Proof. We assume $l_1 \geq l_2$. The other case is similar. First we check the statement for the highest weight elements. In the component $B_{(l_1+d,l_2-d)}$ it is given by

$$\overbrace{\boxed{1}\;\cdots\;\boxed{1}}^{l_1} \otimes \overbrace{\boxed{1}\cdot\cdot\boxed{1}}^{d}\overbrace{\boxed{2}\cdot\cdot\boxed{2}}^{l_2-d}.$$

Correspondingly, the tableau



is indeed nonmovable. Now we have only to show $D(\tilde{f}_i(b_1 \otimes b_2)) = D(b_1 \otimes b_2)$ for $\tilde{f}_i(b_1 \otimes b_2) \neq 0$. Put $s = \varphi_i(b_1)$ and $s' = \varphi_{i+1}(b_2)$. We consider the cases (i) $s > s'$ and (ii) $s \leq s'$ separately. In the case (i), $\tilde{f}_i(b_1 \otimes b_2) = (\tilde{f}_i b_1) \otimes b_2$ and $\tilde{f}_i b_1$ is obtained by changing the rightmost i in b_1 into $i+1$. Thus $D(\tilde{f}_i(b_1 \otimes b_2)) \geq D(b_1 \otimes b_2)$. Suppose $D(\tilde{f}_i(b_1 \otimes b_2)) > D(b_1 \otimes b_2)$. This can happen only in the situation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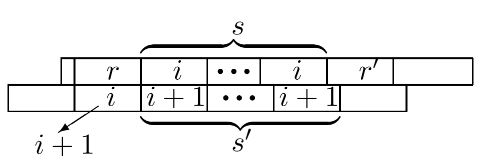
where $i+1 \leq r < r'$. However from $s > s'$ the tableau here for $b_1 \otimes b_2$ is already non-semistandard, which is a contradiction. In the case (ii), $\tilde{f}_i(b_1 \otimes b_2) = b_1 \otimes (\tilde{f}_i b_2)$ and $\tilde{f}_i b_2$ is obtained by changing the rightmost i in b_2 into $i+1$. Thus $D(\tilde{f}_i(b_1 \otimes b_2)) \leq D(b_1 \otimes b_2)$. Suppose $D(\tilde{f}_i(b_1 \otimes b_2)) < D(b_1 \otimes b_2)$. This can happen only in the situation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where $r < i < r'$. In fact only $s = s'$ is possible so that the above tableau for $b_1 \otimes b_2$ becomes semistandard within the constraint $s \leq s'$. But then $D(\tilde{f}_i(b_1 \otimes b_2)) < D(b_1 \otimes b_2)$ implies $r' < i+1$, which is a contradiction. ∎

Let us give another and a more elementary proof of Proposition 2.3 in the homogeneous case $l_1 = l_2 = l$ using the definition of (2.1). For any pair $v_{a_1\ldots a_l}, v_{b_1\ldots b_l} \in B_l$ and an integer $d \in \{0, 1, \ldots, l\}$, let $T_d(v_{a_1\ldots a_l}, v_{b_1\ldots b_l})$ be the tableau below:

$$T_d(v_{a_1\ldots a_l}, v_{b_1\ldots b_l}) = \begin{array}{l} \overbrace{\phantom{\boxed{b_1}\boxed{b_2}\;\;}}^{d}\overbrace{\boxed{a_1}\boxed{a_2}\;\;\boxed{a_{l-d}}\;\;\boxed{a_l}}^{l} \\ \underbrace{\boxed{b_1}\boxed{b_2}\;\;\boxed{b_{d+1}}\;\;\boxed{b_l}}_{l} \end{array}. \tag{2.9}$$

Proposition 2.4. *Let $v_{a_1\ldots a_l}, v_{b_1\ldots b_l} \in B_l$ and $d \in \{0, 1, \ldots, l\}$. Then*
(i) $H_l(v_{a_1\ldots a_l}, v_{b_1\ldots b_l}) \le d$ if and only if the tableau $T_d(v_{a_1\ldots a_l}, v_{b_1\ldots b_l})$ is semistandard.
(ii) $H_l(v_{a_1\ldots a_l}, v_{b_1\ldots b_l}) = d$ if and only if the tableau $T_d(v_{a_1\ldots a_l}, v_{b_1\ldots b_l})$ is nonmovable.

Proof. Suppose the tableau $T_d(v_{a_1\ldots a_l}, v_{b_1\ldots b_l})$ in (2.9) is semistandard. Then $a_i < b_{i+d}$ holds for all $i = 1, \ldots, l-d$. Thus for the permutation

$$\sigma = \begin{pmatrix} 1 & \ldots & l-d & l-d+1 & \ldots & l \\ d+1 & \ldots & l & 1 & \ldots & d \end{pmatrix},$$

$\sum_{i=1}^{l} H_1(v_{a_i}, v_{b_{\sigma(i)}}) = \sum_{i=l-d+1}^{l} H_1(v_{a_i}, v_{b_{\sigma(i)}}) \le d$ holds. Therefore $H_l(v_{a_1\ldots a_l}, v_{b_1\ldots b_l}) \le d$.

Conversely, suppose $H_l(v_{a_1\ldots a_l}, v_{b_1\ldots b_l}) \le d$. Then there is a permutation σ and a subset J of $\{1, 2, \ldots, l\}$ such that $\#J \ge l-d$ and $a_i < b_{\sigma(i)}$ for each $i \in J$. Now if $T_d(v_{a_1\ldots a_l}, v_{b_1\ldots b_l})$ is *not* semistandard, there exists a number $j_0 \le l-d$ such that $a_{j_0} \ge b_{j_0+d}$. It follows that

$$b_1 \le b_2 \le \cdots \le b_{j_0+d} \le a_{j_0} \le a_{j_0+1} \le \cdots \le a_l.$$

This implies $\#(J \cap \{j_0, j_0+1, \ldots, l\}) \le l - j_0 - d$. Thus we have $\#J \le l-d-1$, which contradicts the assumption. It is clear that (ii) is equivalent to (i). $\square$

3. Spectral decomposition

3.1. **Spectrum.** For $\vec{h} = (h_i) \in \{0, 1, \ldots, l\}^{\mathbf{N}}$ let $\tilde{\kappa}(\vec{h})$ be an infinite skew diagram,

$$\tilde{\kappa}(\vec{h}) = \begin{array}{l} \;\overbrace{}^{h_2}\Box\Box\cdots\Box \;\;\cdot^{\cdot^{\cdot}} \\ \overbrace{}^{h_1}\Box\Box\cdots\Box \\ \Box\Box\cdots\Box \end{array},$$

where each row of $\tilde{\kappa}(\vec{h})$ has l boxes. For $\vec{h} \in \{0, 1, \ldots, l\}^{\mathbf{N}}$ and $\vec{s} = (s_i) \in B_l^{\mathbf{N}}$ with $s_i = (v_{a_{i1}\ldots a_{il}})$, we associate a tableaux $T_{\vec{h}}(\vec{s})$ of shape $\tilde{\kappa}(\vec{h})$,

$$T_{\vec{h}}(\vec{s}) = \begin{array}{l} \phantom{\boxed{a_{11}}\boxed{a_{12}}\;}\overbrace{\phantom{\boxed{a_{21}}\boxed{a_{22}}}}^{h_2}\boxed{a_{31}}\boxed{a_{32}}\;\cdots\;\boxed{a_{3l}} \;\;\cdot^{\cdot^{\cdot}} \\ \overbrace{\phantom{\boxed{a_{11}}\boxed{a_{12}}}}^{h_1}\boxed{a_{21}}\boxed{a_{22}}\;\cdots\;\boxed{a_{2l}} \\ \boxed{a_{11}}\boxed{a_{12}}\;\cdots\;\boxed{a_{1l}} \end{array}.$$

We introduce the *local energy map* $\rho : B_l^{\mathbf{N}} \to \{0,1,\dots,l\}^{\mathbf{N}}$ as

$$\rho : \vec{s} = (s_i) \mapsto \vec{h} = (h_i), \quad h_i = H_l(s_{i+1}, s_i). \tag{3.1}$$

Proposition 3.1. *(i) For any $\vec{s} \in B_l^{\mathbf{N}}$, the tableau $T_{\rho(\vec{s})}(\vec{s})$ is nonmovable.*
(ii) The map

$$\tilde{\varphi} : \vec{s} \mapsto T_{\rho(\vec{s})}(\vec{s}).$$

gives a one-to-one correspondence between the elements of $\rho^{-1}(\vec{h})$ and the nonmovable tableaux of shape $\tilde{\kappa}(\vec{h})$.

Proof. (i). Let $\vec{h} = \rho(\vec{s})$. By definition, $T_{h_i}(v_{a_{i+1 1}\dots a_{i+1 l}}, v_{a_{i1}\dots a_{il}})$ in (2.9) is the ith and $i+1$th rows (from the bottom) of $T_{\vec{h}}(\vec{s})$. Then it follows from Proposition 2.4 that $T_{\vec{h}}(\vec{s})$ is nonmovable. (ii). It is clear that $\tilde{\varphi}$ is injective. To see $\tilde{\varphi}$ maps $\rho^{-1}(\vec{h})$ onto the set of the nonmovable tableaux of shape $\tilde{\kappa}(\vec{h})$, suppose T is a nonmovable tableau of shape $\tilde{\kappa}(\vec{h})$. Then there exists a unique $\vec{s} \in B_l^{\mathbf{N}}$ such that $T = T_{\vec{h}}(\vec{s})$. From Proposition 2.4, we have $\rho(\vec{s}) = \vec{h}$. □

The *length* of a skew diagram λ/μ, denoted by $l(\lambda/\mu)$, is defined as

$$l(\lambda/\mu) = \max_i \{\lambda'_i - \mu'_i\},$$

where λ' is the conjugate of λ.

Lemma 3.2. *Let $\vec{h} = (h_i) \in \{0,1,\dots,l\}^{\mathbf{N}}$. The following conditions are equivalent:*
(i) $\vec{h} \in \rho(B_l^{\mathbf{N}})$.
(ii) *There exists a nonmovable tableau of skew shape $\tilde{\kappa}(\vec{h})$.*
(iii) *The length of $\tilde{\kappa}(\vec{h})$ is at most n.*
(iv) $h_i + h_{i+1} + \cdots + h_{i+n-1} \geq l$ *for any* $i \geq 1$.

Proof. (i) ⇔ (ii). This is due to Proposition 3.1.

(ii) ⇔ (iii). Suppose the length of $\tilde{\kappa}(\vec{h})$ is more than n. Then $\tilde{\kappa}(\vec{h})$ has at least one column whose height is greater than n. But, then, $\tilde{\kappa}(\vec{h})$ does not admit any semistandard tableau of its shape (with n numbers). In particular, there is no nonmovable tableau of shape $\tilde{\kappa}(\vec{h})$. Conversely, suppose the length of $\tilde{\kappa}(\vec{h})$ is at most n. Then we can construct a nonmovable tableau of shape $\tilde{\kappa}(\vec{h})$ by filling the boxes of each column of $\tilde{\kappa}(\vec{h})$ from the top to the bottom by the numbers $1, 2, 3, \dots$ in this order. It is clear that the resulting tableau is semistandard. Furthermore, the number in the top box of each column is always 1. Such a tableau is always nonmovable.

(iii) ⇔ (iv). This is clear from the shape of $\tilde{\kappa}(\vec{h})$. □

We define $\mathrm{Sp}_K = \rho(\mathcal{S}_K)$, and call it the *spectrum of $\mathcal{S}_K$*. By definition $\mathrm{Sp_K}$ is a subset of $\rho(B_l^{\mathbf{N}})$. We set $\mathrm{Sp} = \bigsqcup_{K \in \mathcal{K}_l} \mathrm{Sp}_K$.

Lemma 3.3. *Let $K \in \mathcal{K}_l$.*
(i) It holds that $\rho(\vec{s}^{(K)}) = K$. In particular, K belongs to Sp_{K}.
(ii) For any $\vec{s} \in B_l^{\mathbf{N}}$, $\vec{s} \approx \vec{s}^{(K)}$ if and only if $\rho(\vec{s}) \approx K$.

Proof. (i) The tableau $T_{k_i}(s_{i+1}^{(K)}, s_i^{(K)})$ is nonmovable as shown below:

$$
\begin{array}{c|c|c|c|c|}
\multicolumn{1}{c}{\overbrace{}^{k_i}} & \multicolumn{1}{c}{\overbrace{}^{k_{i+1}}} & \multicolumn{3}{c}{} \\
\cline{2-5}
\multicolumn{1}{c|}{} & 1\ \cdots\ 1 & \qquad\qquad & \qquad\quad & n\cdots n \\
\hline
\multicolumn{1}{|c|}{1\cdots 1} & 2\ \cdots\ 2 & & n\cdots n & \multicolumn{1}{c}{} \\
\cline{1-4}
\end{array}\ .
$$

Thus $H(s_{i+1}^{(K)}, s_i^{(K)}) = k_i$. (ii) If $\vec{s} \approx \vec{s}^{(K)}$, then we have $\rho(\vec{s}) \approx \rho(\vec{s}^{(K)}) = K$ by (i). Conversely, let $\vec{h} = \rho(\vec{s})$, and suppose $\vec{h} \approx K$. Then except for finitely many columns the heights of the columns of $\tilde{\kappa}(\vec{h})$ are n. The content in any height-n column of a semistandard tableau is uniquely determined as $1, 2, \ldots, n$ from the top to the bottom. It follows that $\vec{s} \approx \vec{s}^{(K)}$. $\square$

From Lemmas 3.2 and 3.3 it follows that

Theorem 3.4. *An element $\vec{h} = (h_i) \in \{0, 1, \ldots, l\}^{\mathbf{N}}$ belongs to Sp_K if and only if it satisfies the conditions,*

$$\text{(i) } h_i + h_{i+1} + \cdots + h_{i+n-1} \geq l \text{ for any } i > 0. \tag{3.2a}$$

$$\text{(ii) } \vec{h} \approx K. \tag{3.2b}$$

3.2. **Spectral decomposition.** The local energy map ρ induces the *spectral decomposition* of $\mathcal{S}_K$,

$$\mathcal{S}_K = \bigsqcup_{\vec{h} \in \mathrm{Sp}_K} \mathcal{S}_{\vec{h}}, \quad \mathcal{S}_{\vec{h}} := \rho^{-1}(\vec{h}).$$

Let us introduce the character of the degeneracy of the spectrum at $\vec{h}$,

$$\chi_{\vec{h}}(x) = \sum_{\vec{s} \in \mathcal{S}_{\vec{h}}} e^{\mathrm{wt}(\vec{s})}. \tag{3.3}$$

Due to Theorem 2.1, the spectral decomposition of $\mathcal{S}_K$ induces the decomposition of the character of $\mathcal{L}(\Lambda(K))$,

$$\mathrm{ch}\, \mathcal{L}(\Lambda(K))(q, x) = \sum_{\vec{h} \in \mathrm{Sp}_K} q^{\sum_{i=1}^{\infty} i(h_i - k_i)} \chi_{\vec{h}}(x). \tag{3.4}$$

3.3. **Character $\chi_{\vec{h}}$ and finite diagram.** Let us observe that for each element $\vec{h} = (h_i) \in \mathrm{Sp}_K$, except for the case $\vec{h} = K$, there is a unique positive integer J such that $\vec{h}$ is written in the following form

$$\vec{h} = (h_1, \ldots, h_J, (k_{J+1}, \ldots, k_{J+n})^{\infty}), \quad h_J > k_J. \tag{3.5}$$

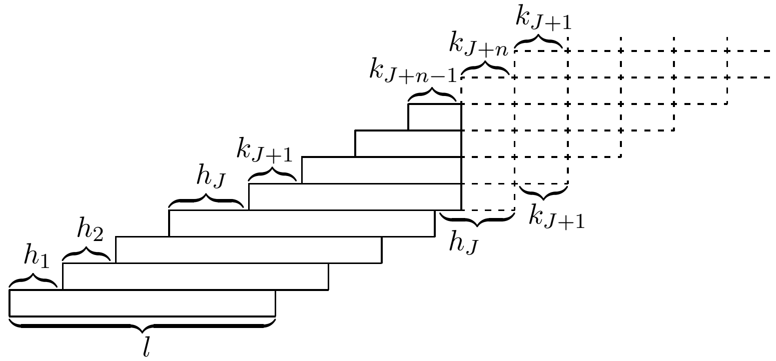


FIGURE 1. A finite diagram $\kappa(\vec{h}) = \kappa(h_1, \dots, h_J)$.

The subsequence $(h_1, \dots, h_J)$ of $\vec{h}$ will be called the *finite part* of $\vec{h}$, and denoted by $\vec{h}_{\text{fin}}$. In the case $\vec{h} = K$, its finite part is defined to be the empty sequence $\emptyset$. For $\vec{h} \in \mathrm{Sp}_K$ with $\vec{h}_{\text{fin}} = (h_1, \dots, h_J)$, let $\kappa(\vec{h}) = \kappa(h_1, \dots, h_J)$ be a finite skew subdiagram of $\tilde{\kappa}(\vec{h})$ as in Fig. 1. The following properties and proposition are easily seen from Fig. 1.

(i) The height of every column of the compliment of $\kappa(\vec{h})$ in $\tilde{\kappa}(\vec{h})$ is n.
(ii) The height of the rightmost column of $\kappa(\vec{h})$ is at most $n-1$.

Proposition 3.5. *There is a one-to-one correspondence between the nonmovable tableaux of shape $\tilde{\kappa}(\vec{h})$ and those of shape $\kappa(\vec{h})$. The correspondence is given by the restriction of a nonmovable tableau of shape $\tilde{\kappa}(\vec{h})$ on $\kappa(\vec{h})$.*

Combining the bijection in Proposition 3.5 with the bijection $\tilde{\varphi}$ in Proposition 3.1, we obtain a bijection

$$\varphi : \mathcal{S}_{\vec{h}} \xrightarrow{\sim} \mathrm{NMT}(\kappa(\vec{h})).$$

Let us introduce the sl_n-weight of a tableau T by $\mathrm{wt}(T) = \sum_{a=1}^{n} m_a \epsilon_a$, where $(m_1, \dots, m_n)$ is the content of T, i.e., m_a is the number counting how many a's are in T.

Proposition 3.6. *The bijection $\varphi : \mathcal{S}_{\vec{h}} \to \mathrm{NMT}(\kappa(\vec{h}))$ is weight-preserving, i.e., for any $\vec{s} \in \mathcal{S}_{\vec{h}}$, $\mathrm{wt}(\vec{s}) = \mathrm{wt}(\varphi(\vec{s}))$ holds.*

Proof. Let $\vec{h}_{\text{fin}} = (h_1, \dots, h_J)$. From Fig. 1, it is clear that for any $\vec{s} = (s_i) \in \rho^{-1}(\vec{h})$, s_i coincides with $s_i^{(K)}$ for any $i \geq J+n$. Thus, by definition,

$$\mathrm{wt}(\vec{s}) = \sum_{i=1}^{n-1} k_i \overline{\Lambda}_i + \sum_{i=1}^{J+n-1} \mathrm{wt}(s_i) - \sum_{i=1}^{J+n-1} \mathrm{wt}(s_i^{(K)}).$$

In the meanwhile

$$\sum_{i=1}^{J+n-1} \mathrm{wt}(s_i) = \mathrm{wt}(\varphi(\vec{s})) - \sum_{i=1}^{n-1} k_{J+i-1}\overline{\Lambda}_i,$$
$$\sum_{i=1}^{J+n-1} \mathrm{wt}(s_i^{(K)}) = \sum_{i=1}^{n-1} k_i\overline{\Lambda}_i - \sum_{i=1}^{n-1} k_{J+i-1}\overline{\Lambda}_i.$$

Therefore we have $\mathrm{wt}(\vec{s}) = \mathrm{wt}(\varphi(\vec{s}))$. □

We define the function

$$t_{\lambda/\mu}(x) = \sum_{T\in \mathrm{NMT}(\lambda/\mu)} e^{\mathrm{wt}(T)}. \tag{3.6}$$

Let $\mathrm{Sp}_{K,\mathrm{fin}}$ denote the set of all the finite parts of $\vec{h}$'s in Sp_K, i.e.,

$$\mathrm{Sp}_{K,\mathrm{fin}} = \{(h_1,\dots,h_J) \mid J \geq 0,\ h_i \in \{0,1,\dots,l\},\ \text{the condition } (*) \text{ when } J \geq 1\},$$

where $(h_1,\dots,h_J) = \emptyset$ for $J = 0$, and the condition $(*)$ is

$$(*)\begin{cases} \sum_{j=0}^{n-1} h_{i+j} \geq l \quad \text{for any } 1 \leq i \leq J-1, \quad (h_{J+i} := k_{J+i},\ 1 \leq i \leq n-1), \\ h_J > k_J. \end{cases}$$

From Proposition 3.6 and (3.4), we have

Theorem 3.7. *(i) The character $\chi_{\vec{h}}$ of $\mathcal{S}_{\vec{h}}$ is equal to the function $t_{\kappa(\vec{h})}$.*
(ii) The character of the level l integrable module $\mathcal{L}(\Lambda(K))$ of $\widehat{sl}_n$ decomposes as

$$\begin{aligned}\mathrm{ch}\,\mathcal{L}(\Lambda(K))(q,x) &= \sum_{\vec{h}\in \mathrm{Sp}_K} q^{\sum_{i=1}^{\infty} i(h_i-k_i)} t_{\kappa(\vec{h})}(x)\\ &= \sum_{(h_1,\dots,h_J)\in \mathrm{Sp}_{K,\mathrm{fin}}} q^{\sum_{i=1}^{J} i(h_i-k_i)} t_{\kappa(h_1,\dots,h_J)}(x)\end{aligned}$$

3.4. **Formula of $t_\kappa(h_1,\dots,h_J)$ by skew Schur functions.** Let $s_{\lambda/\mu} = s_{\lambda/\mu}(x)$ be the *skew Schur function associated to* λ/μ [17]. Let $(h_1,\dots,h_J) \in \mathrm{Sp}_{K,\mathrm{fin}}$ be fixed. We set $I_0 := \{i \in \{1,\dots,J\} | h_i \neq 0\}$ and $\kappa_0 := \kappa(h_1,\dots,h_J)$. For $i_1,\dots,i_p \in I_0$ such that $i_1 < i_2 < \cdots < i_p$, consider a sequence $(h'_1,\dots,h'_J)$, where

$$h'_i = \begin{cases} h_i - 1 & \text{if } i = i_1,\dots,i_p, \\ h_i & \text{otherwise.} \end{cases}$$

Even though $(h'_1,\dots,h'_J)$ does not necessarily belong to $\mathrm{Sp}_{K,\mathrm{fin}}$, we can still define a skew diagram $\kappa(h'_1,\dots,h'_J)$ by Fig. 1. Let $\kappa_{i_1,\dots,i_p} := \kappa(h'_1,\dots,h'_J)$.

Proposition 3.8. *Let* $(h_1, \dots, h_J) \in \mathrm{Sp}_{K,\mathrm{fin}}$. *Then*

$$t_{\kappa(h_1,\dots,h_J)} = s_{\kappa_0} + \sum_{p=1}^{|I_0|} (-1)^p \sum_{\substack{i_1,\dots,i_p \in I_0 \\ i_1 < \cdots < i_p}} s_{\kappa_{i_1,\dots,i_p}}. \tag{3.7}$$

where in the right hand side we impose the relation $x_1 \cdots x_n = 1$.

Example 3.9.

$$t_{\ydiagram{2+2,1+2,2}} = s_{\ydiagram{2+2,1+2,2}} - s_{\ydiagram{1+2,1+2,2}} - s_{\ydiagram{1+2,2,2}} + s_{\ydiagram{2,2,2}}.$$

Proof. For a given skew shape κ, let us denote by $\mathrm{SST}(\kappa)$ the set of all semistandard tableaux of the shape κ. Consider the injection

$$\eta_{i_1,\dots,i_p} : \mathrm{SST}(\kappa_{i_1,\dots,i_p}) \hookrightarrow \mathrm{SST}(\kappa_0)$$

preserving the contents for each row. It is clear that the map $\eta_{i_1,\dots,i_p}$ is weight-preserving. Let $A_{i_1,\dots,i_p}$ denote the image of $\mathrm{SST}(\kappa_{i_1,\dots,i_p})$ in $\mathrm{SST}(\kappa_0)$ under the map. Then we have

(i) $\mathrm{NMT}(\kappa_0) = \mathrm{SST}(\kappa_0) - \bigcup_{i \in I_0} A_i$.

(ii) $A_{i_1,\dots,i_p} = A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_p}$.

From (i),(ii) and the inclusion-exclusion principle we obtain (3.7). ∎

In the case of either $l = 1$ or $n = 2$, the right hand side of the formula (3.7) is written in a simpler form, and reproduces the known results.

Example 3.10. *The case* $l = 1$, n*: general (cf.* [13]*).* The skew diagram $\kappa_0 = \kappa(h_1, \dots, h_J)$ has $I_0 + 1$ columns, and the width of every row of κ_0 is 1. Let $1 \le m_i$ $(i = 1, \dots, I_0 + 1)$ be the height of the ith column (from the left). Let E_m be the mth elementary symmetric polynomial of $x_1, \dots, x_n$ for $m = 0, 1, \dots, n$, and $E_m = 0$ for $m < 0$ and $m > n$. Then

$$s_{\kappa_0} = \prod_{i=1}^{I_0+1} E_{m_i}, \qquad s_{\kappa_{i_1}} = \left(\prod_{i=1}^{j_1-1} E_{m_i} \right) E_{m_{j_1} + m_{j_1+1}} \left(\prod_{i=j_1+2}^{I_0+1} E_{m_i} \right), \qquad \dots,$$

where j_1 is the number determined by $i_1 = m_1 + m_2 + \cdots + m_{j_1}$. Thus the rhs of (3.7) is written as

$$\det_{1 \le i,j \le I_0+1} E_{(\sum_{t=1}^{j} m_t) - (\sum_{t=1}^{i-1} m_t)}.$$

Let $[m_1, \dots, m_{I_0+1}]$ be the *border strip* with $I_0 + 1$ columns such that the ith column (from the left) has height m_i. (A border strip is a connected skew diagram containing no 2×2 block of boxes.) Due to the Jacobi-Trudi formula, the above determinant is equal to the skew Schur function $s_{[m_1,\dots,m_{I_0+1}]}$.

Example 3.11. *The case $n = 2$, l: general (cf. [2]).* The skew diagram $\kappa_0 = \kappa(h_1, \dots, h_J)$ has $J+1$ rows. Let $m_1 = h_1$ and $m_i = h_i + h_{i-1} - l$ for $i = 2, \dots, J+1$. In other words, m_i is the number of the boxes in the ith row of κ_0 without any vertical adjacent boxes. For the simplicity, let us first consider the situation $m_i > 0$ for all i. Let H_m be the mth completely symmetric polynomial of x_1 and x_2. Then

$$s_{\kappa_0} = \prod_{i=1}^{J+1} H_{m_i}, \qquad s_{\kappa_{i_1}} = \left(\prod_{i=1}^{i_1-1} H_{m_i}\right) H_{m_{i_1}-1} H_{m_{i_1+1}-1} \left(\prod_{i=i_1+2}^{J+1} H_{m_i}\right), \qquad \dots$$

By the formulae

$$\begin{aligned} H_{m_1+m_2} &= H_{m_1} H_{m_2} - H_{m_1-1} H_{m_2-1}, \\ H_{m_1+m_2+m_3} &= H_{m_1} H_{m_2} H_{m_3} - H_{m_1-1} H_{m_2-1} H_{m_3} \\ &\quad - H_{m_1} H_{m_2-1} H_{m_3-1} + H_{m_1-1} H_{m_2-2} H_{m_3-1}, \qquad \text{etc.,} \end{aligned}$$

the rhs of (3.7) reduces to $H_{m_1+\dots+m_{J+1}}$. In general the sequence (m_i) is decomposed into $r \ge 1$ components of successive nonzero elements as

$$(m_i) = (m_{11}, \dots, m_{1t_1}, 0, \dots, 0, m_{21}, \dots, m_{2t_2}, 0, \dots, 0, m_{r1}, \dots, m_{rt_r}), \qquad m_{ij} \neq 0.$$

Let $M_i = \sum_{j=1}^{t_i} m_{ij}$. Then the rhs of (3.7) is equal to $\prod_{i=1}^{r} H_{M_i}$.

4. Spectral decomposition and exponents

Let λ be a partition of length $\le m$, μ be a composition, $l(\mu) \le m$, and $d \in \mathbf{Z}_{\ge 0}^{m-1}$ be a sequence of nonnegative integers. In this section we are going to describe a bijection

$$\mathrm{SST}_d(\lambda, \mu) \leftrightarrow \mathrm{LR}_0(\mathrm{Sh}_d(\mu), \lambda) \tag{4.1}$$

between the set of all the semistandard Young tableaux T of shape λ and content μ, with a given set of exponents $d(T) = d = (d_1, d_2, \dots)$, and the set of all the nonmovable Littlewood-Richardson tableaux of (skew) shape $\mathrm{Sh}_d(\mu)$ and content λ. We start with reminding all necessary definitions.

4.1. **Exponents.** ([6, 14, 12]). Let us explain at first a combinatorial definition of the exponents. We start with definition of the descent set $D(T)$ of a given semistandard tableau $T \in \mathrm{SST}(\lambda, \mu)$ of shape λ and content μ. Let indices i and $i+1$ belong to the given tableau T. We say that a pair i and $i+1$ forms a descent in tableau T, if $i+1$ lies strictly below than i in the tableau T. We say that i is a descent of multiplicity ζ_i, if ζ_i is the maximal number such that there exist ζ_i pairs of descents i and $i+1$ in the tableau T with different ends $i+1$. We denote by $D(T)$ a set of all descents with multiplicities in the tableau T.

Example 4.1. Let us take

$$T := \begin{array}{cccccc} 1 & 1 & 2 & 3 & 5 & 5 \\ 2 & 3 & 3 & 4 & & \\ 3 & 4 & 5 & & & \\ 4 & 5 & & & & \end{array} \quad \in \mathrm{SST}((6,4,3,2),(2,2,4,3,4)),$$

then $D(T) = \{1,2,2,3,3,3,4,4\}$ and the multiplicities of descents are $\zeta = (1,2,3,2)$.

Definition 4.2. We define the ith exponent $d_i(T)$ of a semistandard tableau $T \in \mathrm{SST}(\lambda, \mu)$ as follows

$$d_i(T) := \mu_{i+1} - \zeta_i(T), \quad 1 \le i \le m-1.$$

In the Example 4.1 we have $\mu = (2,2,4,3,4)$ and $d = (1,2,0,2)$.

There exists also a group-representation interpretation of the exponents (cf. [6]):

$$d_i(T) = \max_k \{E_{i,i+1}^k \mid T\rangle \ne 0\},$$

where $E_{i,j}$, $1 \le i \ne j \le m$, is a set of generators of the Lie algebra $\mathfrak{gl}_m$, and $|T\rangle$ is the Gelfand-Tsetlin pattern (GT-pattern, for short) corresponding to the tableau T. Here we identify the GT-patterns with a basis in the highest weight λ irreducible representation of $\mathfrak{gl}_m$.

4.2. **Littlewood-Richardson tableaux.** Let A be a skew shape and μ be a composition, $|A| = |\mu|$, $l(\mu) = m$. We denote by $\mathrm{Tab}(A, \mu)$ the set of all the tableaux of shape A and content μ, in other words, the set of all fillings of the shape A by the numbers from 1 to m that have the content μ.

Example 4.3. Let us take $\mu = (8,5,2)$ and $A = (6,6,6,6)/(4,3,2)$. Then

$$\begin{array}{cccccc} & & & & 1 & 1 \\ & & & 1 & 2 & 2 \\ & & 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 \end{array} \in \mathrm{Tab}(A, \mu).$$

For each tableau T we define a word $w(T)$ by reading the numbers of the tableau T along the rows from the right to the left and from the top to the bottom. For the tableau from Example 4.3, we have

$$w(T) = 112213111322211.$$

Definition 4.4. Let us say that a tableau T is the Littlewood-Richardson tableau (LR-tableau, for short), if the corresponding word $w(T)$ is a lattice word and each row of T is semistandard.

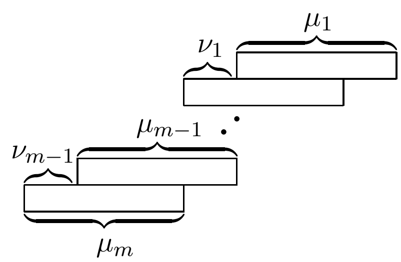
For the reader's convenience, let us remind a definition of a lattice word (see, e.g. [17], p.143):

A word $w = a_1a_2\dots a_N$ in the symbols $1, 2, \dots, m$ is said to be a lattice word if for $1 \le r \le N$ and $1 \le i \le m-1$, the number of occurrences of the symbol i in $a_1a_2\dots a_r$ is not less than the number of occurrences of $i+1$.

In Example 4.3, the tableau T is an LR-tableau.

4.3. **Bijection θ_d.** Let $\mu = (\mu_1, \dots, \mu_m)$ and $\nu = (\nu_1, \dots, \nu_{m-1})$ be compositions. We define a skew shape $\mathrm{Sh}_\nu(\mu)$ as

$\mathrm{Sh}_\nu(\mu) =$



.

In the sequel we will assume that $\nu_i + \mu_i \ge \mu_{i+1}$ holds for any i. Now we are ready to define a map

$$\theta_\nu : \mathrm{SST}(\lambda, \mu) \to \mathrm{Tab}(\mathrm{Sh}_\nu(\mu), \lambda) \tag{4.2}$$

in the following way: Consider a given semistandard tableau $T \in \mathrm{SST}(\lambda, \mu)$, let us fill the kth row of the shape $\mathrm{Sh}_\nu(\mu)$ by numbers $1, 2, \dots,$ according to the indices of the rows in the tableau T which contain k, starting from the top to the bottom.

Example 4.5. Let us take

$$T = \begin{array}{cccccccc} 1 & 1 & 2 & 3 & 3 & 3 & 4 & 4 \\ 2 & 2 & 4 & 4 & 4 & & & \\ 3 & 4 & & & & & & \end{array},$$

and $\nu = (1, 1, 2)$. Then

$$\theta_\nu(T) := \begin{array}{cccccc} & & & & 1 & 1 \\ & & & 1 & 2 & 2 \\ & & 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 \end{array}.$$

The following Lemma is well-known.

Lemma 4.6. *Let T be a semistandard tableau of shape λ, content μ, and $\nu \in \mathbf{Z}_{\ge 0}^{m-1}$ be a composition. Then $\theta_\nu(T)$ is an LR-tableau, and θ_ν defines a bijection*

$$\theta_\nu : \mathrm{SST}(\lambda, \mu) \leftrightarrow \mathrm{LR}(\mathrm{Sh}_\nu(\mu), \lambda), \tag{4.3}$$

where $\mathrm{LR}(A, \eta)$ is the set of all the LR-tableaux (not necessarily semistandard!) of shape A and content η.

In other words, a tableau T is semistandard if and only if for some composition $\nu \in \mathbf{Z}_{\geq 0}^{m-1}$, the tableau $\theta_\nu(T)$ is an LR-tableau. It is easy to see that in general, a tableau $\theta_\nu(T)$, $T \in \mathrm{SST}(\lambda,\mu)$, is not necessarily a semistandard one.

Lemma 4.7. *Let T be a semistandard tableau of shape λ and content μ, and $\nu \in \mathbf{Z}_{\geq 0}^{m-1}$ be a composition satisfying $\nu_i + \mu_i \geq \mu_{i+1}$. Then the tableau $\theta_\nu(T)$ is semistandard if and only if*

$$\nu_i \geq d_i(T) \qquad \text{for any } 1 \leq i \leq m-1. \tag{4.4}$$

Proof. Suppose $\theta_\nu(T)$ is semistandard. The overlapping length between the ith and the $i+1$th rows in $\theta_\nu(T)$ is $s = \mu_{i+1} - \nu_i$. Let $k_1, \ldots, k_s$ $(m_1, \ldots, m_s)$ be the entries of the first (last) s boxes in the ith $(i+1\text{th})$ row. For each pair (k_j, m_j) one can assign a descent pair in T with i $(i+1)$ at the k_jth (m_jth) row therein. This means $\zeta_i \geq s = \mu_{i+1} - \nu_i$. The converse is similar. □

It follows from Lemma 4.7 that

Corollary 4.8. *For a given semistandard tableau T and a composition $\nu \in \mathbf{Z}_{\geq 0}^{m-1}$, the tableau $\theta_\nu(T)$ is nonmovable (see Definition 2.2) if and only if*

$$\nu_i = d_i(T) \qquad \text{for any } 1 \leq i \leq m-1.$$

Before stating our main result of Section 4, let us introduce two additional notations.

The first one is: Given a partition λ, $l(\lambda) \leq m$, and compositions μ, $l(\mu) \leq m$, and d, $l(d) \leq m-1$, we denote by $\mathrm{SST}_d(\lambda,\mu)$ the set of all the semistandard tableaux T of shape λ and content μ such that $d_i(T) = d_i$, $1 \leq i \leq m-1$. The second one is: Given a skew shape A and a partition λ, we denote by $\mathrm{LR}_0(A,\lambda)$ the set of all the nonmovable LR-tableaux of (skew) shape A and content λ. See Definitions 2.2 and 4.4.

Combining together Lemmas 4.6, 4.7 and Corollary 4.8, we obtain the following result which is the main result of Section 4.

Theorem 4.9. *Let λ, μ and d as above, then the map θ_d defines a bijection*

$$\theta_d : \mathrm{SST}_d(\lambda,\mu) \leftrightarrow \mathrm{LR}_0(\mathrm{Sh}_d(\mu),\lambda). \tag{4.5}$$

Corollary 4.10. ("Spectral decomposition" of the Kostka numbers). *Let λ be a partition, and μ be a composition. Then*

$$|\mathrm{SST}(\lambda,\mu)| = \dim V_\lambda(\mu) = K_{\lambda,\mu} = \sum_{d=(d_1,\ldots,d_{m-1}) \in \mathbf{Z}_{\geq 0}^{m-1}} |\mathrm{LR}_0(\mathrm{Sh}_d(\mu),\lambda)|. \tag{4.6}$$

Let us say a few words about the set $\mathrm{LR}_0(\alpha/\beta,\lambda)$. Let $c_{\lambda\mu}^{\nu}$ be the Littlewood-Richardson coefficient, that is the multiplicity of the irreducible highest weight ν representation V_ν of the Lie algebra $\mathfrak{gl}_m$ in the tensor product $V_\lambda \otimes V_\mu$.

Proposition 4.11. *Let α, β be partitions, $\beta \subset \alpha$. Then*

$$|\mathrm{LR}_0(\alpha/\beta, \lambda)| = \sum_{\alpha'/\beta'} (-1)^{\mathrm{sign}(\alpha/\beta,\alpha'/\beta')} c^{\alpha'}_{\beta'\lambda},$$

where the sum extends over all the skew diagrams α'/β' such that $\alpha'_i - \beta'_i = \alpha_i - \beta_i$, $\beta'_i = \beta_i$, or $\beta_i - 1$ for all i, and $\mathrm{sign}(\alpha/\beta, \alpha'/\beta') := |\{i \mid \beta'_i = \beta_i - 1\}|$.

Proof. There is a one-to-one correspondence between the semistandard LR-tableaux of shape α/β and content λ which are movable to those of shape α'/β', and the semistandard LR-tableaux of shape α'/β' and content λ. It is well-known (see, e.g. [17], p.143) that the number of the last tableaux is equal to the Littlewood-Richardson coefficient $c^{\alpha'}_{\beta'\lambda}$. To finish the proof, let us apply the inclusion-exclusion principle. □

Corollary 4.12. *Let A be a skew diagram. Let us denote by $\widehat{A}$ the diagram which is obtained from the diagram A by the 180 degree rotation. Then*

$$|\mathrm{LR}_0(A, \nu)| = |\mathrm{LR}_0(\widehat{A}, \nu)|. \tag{4.7}$$

Proof. Assume that $A = \lambda/\mu$, where $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_m)$. It is easy to see that $\widehat{A} = \widehat{\mu}/\widehat{\lambda}$, where $\widehat{\lambda} = (\lambda_1 - \lambda_m, \lambda_1 - \lambda_{m-1}, \ldots, \lambda_1 - \lambda_2, 0)$ and $\widehat{\mu} = (\lambda_1 - \mu_m, \lambda_1 - \mu_{m-1}, \ldots, \lambda_1 - \mu_1)$. Now we are going to use the following well-known (see, e.g. [12]) symmetry property of the Littlewood-Richardson coefficients:

$$c^{\lambda}_{\mu\nu} = c^{\widehat{\mu}}_{\widehat{\lambda}\nu}. \tag{4.8}$$

From (4.8) and Proposition 4.11 we obtain the equality (4.7). □

In Section 5 we are going to describe a q-analog of (4.6) in a particular case when μ is a rectangular partition (l^m).

5. New combinatorial formula for Kostka–Foulkes polynomials

In Section 5 we give a new combinatorial formula for the Kostka-Foulkes polynomials $K_{\lambda,\mu}(q)$ in the case when μ is a rectangular partition. For the reader's convenience, we remind the basic definitions and results concerning the Kostka-Foulkes polynomials. The proofs and further details can be found in [17].

5.1. **Kostka-Foulkes polynomials.** Let λ and ν be partitions such that $\nu_i \le \lambda_i$ for any i, and $l(\lambda/\nu) \le m$. Let $x = (x_1, \ldots, x_m)$ be set of independent variables. The connection coefficients between the skew Schur functions $s_{\lambda/\nu}(x)$ and the monomial symmetric functions $m_\mu(x)$ are called the (skew) Kostka numbers:

$$s_{\lambda/\nu}(x) = \sum_{\mu} K_{\lambda/\nu,\mu} m_\mu(x). \tag{5.1}$$

It is well-known that the Kostka number $K_{\lambda/\nu,\mu}$ is equal to the number of all the semistandard tableaux of shape λ/ν and content μ.

Definition 5.1. The Hall-Littlewood function $P_\lambda(x;q)$ corresponding to a partition λ is defined by the following formula

$$P_\lambda(x;q) = c_{\lambda,m}(q) \sum_{w\in S_m} w\left(x^\lambda \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j}\right), \tag{5.2}$$

where $c_{\lambda,m}(q)$ is a normalization constant defined by

$$c_{\lambda,m}(q) = v_{m-l(\lambda)}(q)/v_\lambda(q),$$

$$v_\lambda(q) = \prod_{i\geq 1} \prod_{j=1}^{\lambda'_i-\lambda'_{i+1}} \frac{1-q^j}{1-q}.$$

Definition 5.2. The Kostka-Foulkes polynomials $K_{\lambda/\nu,\mu}(q)$ are defined as the connection coefficients between the skew Schur and Hall-Littlewood functions:

$$s_{\lambda/\nu}(x) = \sum_\mu K_{\lambda/\nu,\mu}(q) P_\mu(x;q). \tag{5.3}$$

A combinatorial description of the Kostka-Foulkes polynomials $K_{\lambda,\mu}(q)$ has been found by A. Lascoux and M.-P. Schützenberger [16], and goes as follows. Given two partitions λ and μ, it is possible to attach to each semistandard tableau $T \in \mathrm{SST}(\lambda,\mu)$ a positive integer $c(T)$ (charge of tableau T) such that

$$K_{\lambda,\mu}(q) = \sum_{T\in \mathrm{SST}(\lambda,\mu)} q^{c(T)}. \tag{5.4}$$

Below we give a definition of the charge of a tableau, according to A. Lascoux and M.-P. Schützenberger, [16, 17]. Let λ and μ be partitions, and let $T \in \mathrm{SST}(\lambda,\mu)$. Consider the word $w(T)$ which corresponds to the tableaux T, see [17], Chapter I, §9, or our Section 4.2. The charge $c(T)$ of the tableau T is defined as the charge of corresponding word $w(T)$. Now we define the charge of a word w. Recall that the weight μ of a word w is a sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$, where μ_i is the number of i's occurring in the word w. We assume that the weight μ of a word w is dominant, in other words, $\mu_1 \geq \mu_2 \cdots \geq \mu_N$.

i) First, we assume that w is a standard word (i.e. its weight is $\mu = (1^N)$). We attach an index to each element of w as follows: the index of 1 is equal to 0, and if the index of k is i then the index of $k+1$ is either i or $i+1$ according as it lies to the right or left of k. The charge $c(w)$ of w is defined to be the sum of the indices.

ii) Now assume that w is a word of weight μ and μ is a partition. We extract a standard subword from w in the following way. Reading w from the left to the right, we choose the first occurrence of 1, then the first occurrence of 2 to the right of the 1 chosen and so on. If at some step there is no $s+1$ to the right of the s chosen before, we come back to the beginning of the word. This operation extracts from w a standard subword w_1. Then we erase the word w_1 from w and repeat the procedure to obtain a standard subword w_2, etc.

The charge of w is defined as the sum of the charges of the standard subwords obtained in this way: $c(w) = \sum c(w_i)$. We note that the charge of a word w is zero if and only if the word w is a lattice word.

Example 5.3. Consider a word

$$w = 64\underline{3}2\underline{1}1115\underline{4}33\underline{2}26\underline{5}54\underline{6}. \tag{5.5}$$

The subword w_1 is 314256, consisting of the underlined symbols in w in (5.5). When w_1 is erased, we are left with

$$\widetilde{w} = 642\underline{1}11\underline{5}\underline{3}3\underline{2}\underline{6}5\underline{4}. \tag{5.6}$$

The subword w_2 is 153264, consisting of the underlined symbols in $\widetilde{w}$ in (5.6). When w_2 is erased, we are left with

$$6421135, \tag{5.7}$$

so that $w_3 = 642135$ and $w_4 = 1$.

Now let us compute the charges of the standard subwords w_1, w_2, w_3, w_4. For the word w_1 the indices (attached as subscripts) are $3_1 1_0 4_1 2_0 5_1 6_1$, so that $c(w_1) = 4$; $1_0 5_2 3_1 2_0 6_2 4_1$ for w_2, so that $c(w_2) = 6$; $6_3 4_2 2_1 1_0 3_1 5_2$ for w_3, so that $c(w_3) = 9$; 1_0 for w_4, so that $c(w_4) = 0$: hence $c(w_4) = 4 + 6 + 9 + 0 = 19$.

5.2. **Spectral decomposition of Kostka-Foulkes polynomials.** Let λ, μ be partitions of lengths $\leq m$, and d be a composition, $l(d) \leq m-1$. We define the charge of a tableau $T \in \mathrm{Tab}(\mathrm{Sh}_d(\mu), \lambda)$ as follows

$$c(T) = \sum_{i=1}^{m-1} (m-i) d_i := c(d). \tag{5.8}$$

In general, the map θ_d does not preserve the charges. However, it is well-known (see, e.g. [15, 10], that the map θ_d does preserve the charges, if $\mu = (l^m)$ is a rectangular partition.

Theorem 5.4. *Let $\mu = (l^m)$ be a rectangular partition, then*

$$K_{\lambda,\mu}(q) = \sum_{d=(d_1,\ldots,d_{m-1}) \in \mathbf{Z}_{\geq 0}^{m-1}} q^{c(d)} |\mathrm{LR}_0(\mathrm{Sh}_d(\mu), \lambda)|. \tag{5.9}$$

Example 5.5. Let us consider $\lambda = (4321)$ and $\mu = (2^5)$. It is easy to see that $K_{\lambda,\mu} = 24$. The Bethe ansatz method (see [10]) gives the following expression for the Kostka-Foulkes polynomial

$$\begin{aligned} K_{\lambda,\mu}(q) &= q^3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + q^5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ &= q^3(1 + 2q + 3q^2 + 5q^3 + 5q^4 + 4q^5 + 3q^6 + q^7). \end{aligned}$$

Now, let us apply the spectral decomposition method, Theorem 5.4. In our example there exist 17 terms in the RHS of (5.9). We give a complete list of the tableaux (with their charges) from the disjoint union $\bigsqcup_{d\in\mathbf{Z}_{\geq 0}^{m-1}} \mathrm{LR}_0(\mathrm{Sh}_d(\mu),\lambda)$.

$$\begin{array}{cccccc}
\begin{array}{ccccc} &&1&1\\ &1&2\\ 1&2\\ 2&3\\ 3&4 \end{array} &
\begin{array}{ccccc} &&1&1\\ &&2&2\\ &&3&3\\ &1&4\\ 1&2 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &1&1\\ &2&2\\ &3&3\\ 2&4 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&1&2\\ &&2&3\\ &&3&4\\ 1&2 \end{array} &
\begin{array}{ccccc} &&1&1\\ &&2&2\\ 1&1\\ 2&3\\ 3&4 \end{array} &
\begin{array}{ccccc} &&1&1\\ &&2&2\\ &&3&3\\ 1&1\\ 2&4 \end{array} \\
\\
7 & 3 & 9 & 6 & 6 & 4
\end{array}$$

$$\begin{array}{cccccc}
\begin{array}{ccccc} &&&1&1\\ &&1&2\\ &1&3\\ 2&2\\ 3&4 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&1&2\\ &2&2\\ 1&3\\ 3&4 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&&2&2\\ &&1&3\\ &1&4\\ 2&3 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&&2&2\\ &&1&3\\ &2&3\\ 1&4 \end{array} &
\begin{array}{ccccc} &&1&1\\ &&2&2\\ &1&3\\ 1&2\\ 3&4 \end{array} &
\begin{array}{ccccc} &&1&1\\ &&2&2\\ &1&3\\ 1&3\\ 2&4 \end{array} \\
\\
9 & 9 & 6 & 6 & 5 & 5
\end{array}$$

$$\begin{array}{cccccc}
\begin{array}{ccccc} &&&1&1\\ &&1&2\\ &1&2\\ &3&3\\ 2&4 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&1&2\\ &1&3\\ &2&4\\ 2&3 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&1&2\\ &2&2\\ &3&3\\ 1&4 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&1&2\\ &&2&3\\ &1&4\\ 2&3 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&1&2\\ &&2&3\\ &2&3\\ 1&4 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&1&2\\ &&2&3\\ &2&4\\ 1&3 \end{array} \\
\\
8 & 8 & 8 & 7 & 7 & 7
\end{array}$$

$$\begin{array}{cccccc}
\begin{array}{ccccc} &&&1&1\\ &1&1\\ &2&2\\ 2&3\\ 3&4 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&&2&2\\ &&1&3\\ &&3&4\\ 1&2 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&1&2\\ &&2&3\\ 1&2\\ 3&4 \end{array} &
\begin{array}{ccccc} &&&1&1\\ &&&2&2\\ &1&1\\ &3&3\\ 2&4 \end{array} &
\begin{array}{ccccc} &&1&1\\ &1&2\\ &2&3\\ 1&3\\ 2&4 \end{array} &
\begin{array}{ccccc} &&1&1\\ &&2&2\\ &1&3\\ &2&4\\ 1&3 \end{array} \\
\\
10 & 5 & 8 & 7 & 6 & 4
\end{array}$$

Hence, the spectral decomposition of the Kostka-Foulkes polynomial $K_{(4321),(2^5)}(q)$ has the following form

$$q^3+(1+1)q^4+(2+1)q^5+(1+2+1+1)q^6+(1+3+1)q^7+(1+3)q^8+(1+2)q^9+q^{10}.$$

One can recognize a certain symmetry among the tableaux from the above list. This symmetry is a consequence of the Schützenberger involution action on the set $\mathrm{SST}(\lambda,\mu)$. More precisely, it follows from [10], Corollary 4.2, that if we denote by S the Schützenberger involution

$$S:\ \mathrm{SST}(\lambda,(l^m)) \to \mathrm{SST}(\lambda,(l^m)),$$

then we have the following equality for the exponents:

$$d(S(T)) = \overleftarrow{d}\,(T),$$

where for any sequence $\alpha=(\alpha_1,\ldots,\alpha_m)$ we set $\overleftarrow{\alpha} := (\alpha_m,\alpha_{m-1},\ldots,\alpha_2,\alpha_1)$. On the other hand, it follows from Corollary 4.12 that

$$|\mathrm{LR}_0(\mathrm{Sh}_d(\mu),\lambda)| = |\mathrm{LR}_0(\mathrm{Sh}_{\overleftarrow{d}}(\mu),\lambda)|.$$

If we define $\mathrm{ind}(d)=\sum_{j=1}^{m-1} jd_j$ (cf. [10, 14]), we obtain the following

Corollary 5.6. *Let $\mu=(l^m)$ be a rectangular partition, then*

$$K_{\lambda,\mu}(q) = \sum_{d=(d_1,\ldots,d_{m-1})\in\mathbf{Z}_{\geq 0}^{m-1}} q^{\mathrm{ind}(d)}|\mathrm{LR}_0(\mathrm{Sh}_d(\mu),\lambda)|.$$

6. Truncated characters and branching functions

6.1. **Kostka–Foulkes polynomials and the truncated characters.** We shall explain how our main theorems, Theorems 3.7 and 5.4, are related to each other. Let us consider the special situation of (2.4) where $\Lambda(K)=l\Lambda_k$, $(k=0,\ldots,n-1)$. The configuration space $\mathcal{S}_k := \mathcal{S}_K$ admits a filtration, $\mathcal{S}_{k,k}\subset\mathcal{S}_{k,n+k}\subset\mathcal{S}_{k,2n+k}\subset\cdots\subset\mathcal{S}_k$, where

$$\mathcal{S}_{k,m} = \{(s_1,\ldots,s_m,(v_{n\ldots n},v_{n-1\ldots n-1},\ldots,v_{1\ldots 1})^\infty)\},\qquad m\equiv k \bmod n.$$

Each element of $\mathcal{S}_{k,m}$ is naturally identified with a spin configuration on a finite lattice of size m by the truncation $(s_i)_{i=1}^\infty \mapsto (s_i)_{i=1}^m$. For a tableau T we define the $\mathfrak{gl}_n$-weight of T as $\overline{\mathrm{wt}}(T) := \sum_{a=1}^n m_a\overline{\epsilon}_a$, where $(m_1,\ldots,m_n)$ is the content of T and $\overline{\epsilon}_1,\ldots,\overline{\epsilon}_n$ are linearly independent vectors. We introduce the *truncated character* $F_{k,m}$ of $\mathcal{S}_k$,

$$F_{k,m}(q,x) = q^{A_{k,m}} \sum_{\vec{s}\in\mathcal{S}_{k,m}} q^{E(\vec{s})} e^{\sum_{i=1}^m \overline{\mathrm{wt}}(s_i)},$$

$$A_{k,m} = \frac{1}{2}lnN(N-1)+lNk,\quad N=\frac{1}{n}(m-k),$$

where $x_1=e^{\overline{\epsilon}_1},\ldots,x_n=e^{\overline{\epsilon}_n}$ are independent variables here. By definition, it has the property,

$$\mathrm{ch}\,\mathcal{L}(l\Lambda_k)(q,x) = \lim_{\substack{m\to\infty\\ m\equiv k \bmod n}} q^{-A_{k,m}}\ F_{k,m}(q,x)\big|_{x_1\cdots x_n=1}. \tag{6.1}$$

Here we used the fact that $H_l(v_{n\dots n}, s)(= l)$ is independent of $s \in B_l$. The following theorem states that the branching functions of the truncated character are the Kostka–Foulkes polynomials.

Theorem 6.1 ([18])**.**

$$F_{k,m}(q,x) = \sum_{\lambda} K_{\lambda,(l^m)}(q) s_\lambda(x), \qquad m \equiv k \bmod n, \tag{6.2}$$

where the sum extends over the partitions λ of lm such that $l(\lambda) \le n$.

The spectral decomposition of $\operatorname{ch} \mathcal{L}(l\Lambda_k)$ (Theorem 3.7) induces that of $F_{k,m}$,

$$F_{k,m}(q,x) = \sum_{d=(d_1,\dots,d_{m-1})} q^{\sum_{i=1}^{m-1}(m-i)d_i} \overline{t}_{\mathrm{Sh}_d((l^m))}(x), \quad \overline{t}_{\lambda/\mu}(x) := \sum_{T \in \mathrm{NMT}(\lambda/\mu)} e^{\overline{\mathrm{wt}}(T)}. \tag{6.3}$$

Furthermore, the subcharacter $\overline{t}_{\mathrm{Sh}_d((l^m))}$ is expanded by the Schur functions as

$$\overline{t}_{\mathrm{Sh}_d((l^m))} = \sum_{\lambda} |\mathrm{LR}_0(\mathrm{Sh}_d((l^m)), \lambda)| s_\lambda \tag{6.4}$$

thanks to Propositions 3.8 and 4.11. By comparing the formulae (6.2)–(6.4), we see that Theorem 5.4 is equivalent to Theorem 6.1.[1] In other words, we can interpret Theorem 5.4 as the decomposition of the branching functions $K_{\lambda,(l^m)}(q)$ induced from the spectral decomposition of the truncated character $F_{k,m}$.

Below we remark the relation between our bijection,

$$\theta : \mathrm{SST}(\lambda, (l^m)) \longrightarrow \bigsqcup_d \mathrm{LR}_0(\mathrm{Sh}_d((l^m)), \lambda), \qquad \theta|_{\mathrm{SST}_d(\lambda,(l^m))} = \theta_d, \tag{6.5}$$

and the bijection by Nakayashiki and Yamada [18],

$$\pi : \mathrm{SST}(\lambda, (l^m)) \longrightarrow (B_l^m)_\lambda^{\mathrm{high}},$$

$$(B_l^m)_\lambda^{\mathrm{high}} := \{\vec{s} = (s_1, \dots, s_m) \in B_l^m \mid \sum_{i=1}^m \mathrm{wt}(s_i) = \lambda, \quad \tilde{e}_i(s_1 \otimes \cdots \otimes s_m) = 0\}.$$

Here $\pi(T) = (s_1, \dots, s_m)$ is defined so that the contents of s_i are the indices of the rows in T which contain i. For example,

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 & \\ \cline{1-3} 3 & 3 & & \\ \cline{1-2} \end{array} \mapsto \Big(s_1 = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array}, s_2 = \begin{array}{|c|c|c|} \hline 2 & 2 & 2 \\ \hline \end{array}, s_3 = \begin{array}{|c|c|c|} \hline 1 & 3 & 3 \\ \hline \end{array} \Big).$$

Let

$$(B_l^m)_\lambda := \{\vec{s} = (s_1, \dots, s_m) \in B_l^m \mid \sum_{i=1}^m \mathrm{wt}(s_i) = \lambda\},$$

$$\mathrm{NMT}(\mathrm{Sh}_d((l^m)), \lambda) := \{T \in \mathrm{NMT}(\mathrm{Sh}_d((l^m))) \mid \text{the content of } T \text{ is } \lambda\}.$$

[1]Thus we have given an alternative proof of Theorem 6.1. Conversely, we have our second proof of Theorem 5.4 using Theorem 6.1.

We have a natural bijection

$$\Phi : (B_l^m)_\lambda \longrightarrow \bigsqcup_d \mathrm{NMT}(\mathrm{Sh}_d((l^m)),\lambda) \tag{6.6}$$

such that the content of the ith row of $\Phi(s_1,\dots,s_m)$ equals to that of s_i; for example,

$$\Phi : \left(\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1&3&3\\\hline\end{array} \right) \mapsto \begin{array}{cccc}\cline{2-4} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1}\\ \cline{2-4} & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2}\\ \cline{1-4} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} & \\ \cline{1-3}\end{array}\,.$$

Then

Proposition 6.2. *The diagram*

$$\begin{tikzcd} & (B_l^m)_\lambda^{\mathrm{high}} \arrow[r, phantom, "\subset"] & (B_l^m)_\lambda \arrow[d, "\simeq\ \Phi"] \\ \mathrm{SST}(\lambda,(l^m)) \arrow[ur, "\pi\ \simeq"] \arrow[dr, "\theta\ \simeq"'] & & \\ & \bigsqcup_d \mathrm{LR}_0(\mathrm{Sh}_d((l^m)),\lambda) \arrow[r, phantom, "\subset"] & \bigsqcup_d \mathrm{NMT}(\mathrm{Sh}_d((l^m)),\lambda) \end{tikzcd}$$

is commutative.

Corollary 6.3. Φ *gives a bijection between* $(B_l^m)_\lambda^{\mathrm{high}}$ *and* $\bigsqcup_d \mathrm{LR}_0(\mathrm{Sh}_d((l^m)),\lambda)$.

6.2. **Branching functions as limits of the Kostka–Foulkes polynomials.** As an application of Theorem 6.1 we can identify a limit of the Kostka–Foulkes polynomial $K_{\lambda,(l^m)}(q)$ with a branching function $b_\lambda^\Lambda(q)$ of an $\widehat{sl}_n$-module:

$$\mathrm{ch}\,\mathcal{L}(\Lambda)(q,x) = \sum_\lambda b_\lambda^\Lambda(q)\ s_\lambda(x)\big|_{x_1\cdots x_n=1}\,, \tag{6.7}$$

where the sum extends over all partitions λ such that $l(\lambda) \le n-1$, and $|\lambda| \equiv |\Lambda| \mod n$. By comparing (6.1), (6.2) and (6.7), we have

Corollary 6.4 ([18],[11])**.** *Let* $0 \le k \le n-1$*. Then*

$$b_\lambda^{l\Lambda_k}(q) = \lim_{N\to\infty} q^{-A_{k,k+Nn}} K_{\lambda_N,\mu_N}(q), \tag{6.8}$$

where $\lambda_N = \lambda + \left(\left(\frac{l(k+Nn)-|\lambda|}{n}\right)^n\right)$ *and* $\mu_N = (l^{k+Nn})$.

The formula (6.8) was originally conjectured in [11] for $k=0$ and proved in [18]. A similar formula to Corollary 6.4 for the branching functions b_λ^Λ for a general level l dominant integral weight $\Lambda \ne l\Lambda_k$ can be obtained, as we describe below, by considering the truncated character of the space $\mathcal{S}_K$, $\Lambda = \Lambda(K)$. In the general case, however, the polynomials appearing in the right hand side of (6.8) are no longer identified with the Kostka–Foulkes polynomials.

For a given semistandard tableau of skew shape $T \in \mathrm{SST}(\lambda/\nu,\mu)$, $l(\mu) \le m$, we define the (extended) exponents $d_0(T), d_1(T), \dots, d_{m-1}(T)$, where the definition of $d_i(T)$, $i>0$ remains the same as in Definition 4.2, and $d_0(T) := \mu_1 - \zeta_0(T)$ with

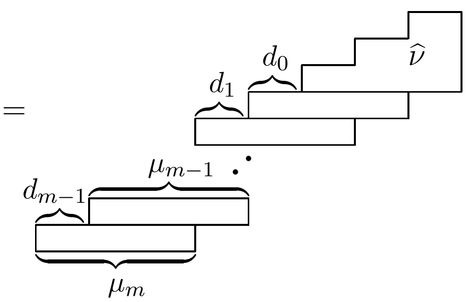
$\zeta_0(T)$ the number of 1's in T which does *not* belong to the first row. Notice that, when λ/ν is a Young diagram, i.e., $\nu = \emptyset$, $d_0(T) = \mu_1$ is independent of T for given λ and μ. Let $\mathrm{SST}_d(\lambda/\nu, \mu)$ be the set of all semistandard tableaux $T \in \mathrm{SST}(\lambda/\nu, \mu)$ such that $d_i(T) = d_i$, $0 \le i \le m-1$. Theorem 4.9 is extended to skew shape tableaux as

Theorem 6.5. *There is a natural bijection*

$$\theta_d : \mathrm{SST}_d(\lambda/\nu, \mu) \leftrightarrow \mathrm{LR}_0(\mathrm{Sh}_d(\mu, \nu), \lambda), \tag{6.9}$$

where $d = (d_0, d_1, \dots, d_{m-1})$,

$\mathrm{Sh}_d(\mu, \nu) =$



,

and $\widehat{\nu}$ *is the skew diagram obtained from* ν *by the 180 degree rotation.*

The bijection θ_d is defined as follows: For a given $T \in \mathrm{SST}_d(\lambda/\nu, \mu)$, the contents of the $l(\nu) + k$th row in $\theta_d(T)$ are given by the indices of the rows in T which contain k (the contents of the first $l(\nu)$ rows in $\theta_d(T)$ are uniquely determined from the semistandardness and the LR property of $\theta_d(T)$). The theorem is easily derived from Theorem 4.9, and it includes Theorem 4.9 as the special case when $\nu = \emptyset$.

Keeping Theorem 6.5 in mind, we introduce two functions. Firstly, for a skew diagram λ/ν and a composition μ with $|\lambda/\nu| = |\mu|$, $m = l(\mu)$, we define a polynomial,

$$G_{\lambda/\nu,\, \mu}(q) := \sum_{T \in \mathrm{SST}(\lambda/\nu,\, \mu)} q^{\sum_{i=0}^{m-1}(m-i)d_i(T)}.$$

Secondly, for $K = (k_i) \in \mathcal{K}_l$ and a positive integer m, we define the truncated character $F_{K,m}$ of $\mathcal{S}_K$ as follows: Consider a filtration $\mathcal{S}_{K,1} \subset \mathcal{S}_{K,2} \subset \cdots \subset \mathcal{S}_K$, where

$$\mathcal{S}_{K,m} := \{(s_i) \mid H(s_{i+1}, s_i) = k_i \text{ for any } i \ge m+1\}.$$

For $\vec{s} \in \mathcal{S}_{K,m}$, $s_i = s_i^{(K)}$ holds for any $i \ge m+n$, but not necessarily so for $i \le m+n-1$ (cf. Fig. 1). Now we define

$$F_{K,m}(q, x) := q^{B_{K,m}} \left(\prod_{i=2}^{n} x_i^{-w_i} \right) \sum_{\vec{s} \in \mathcal{S}_{K,m}} q^{E(\vec{s})} e^{\sum_{i=1}^{m+n-1} \overline{\mathrm{wt}}(s_i)},$$

$$B_{K,m} = \sum_{i=1}^{m} i k_i, \qquad w_i = \sum_{j=1}^{i} k_{m+j-1}.$$

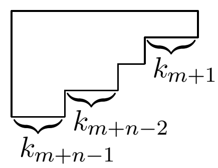
Then, due to the spectral decomposition of $\mathcal{S}_K$ and Theorem 4.9, we have a generalization of Theorem 6.1.

Theorem 6.6.

$$F_{K,m}(q,x) = \sum_{\lambda} G_{\lambda/\nu_{K,m},(l^m)}(q) s_\lambda(x),$$

where

$$\nu_{K,m} =$$



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and the sum extends over the partitions λ *of* $lm + |\nu_{K,m}|$ *such that* $l(\lambda) \leq n$.

Notice that $\nu_{K,m} = \emptyset$ if and only if $\Lambda(K) = l\Lambda_k$, and $m \equiv k \mod n$ for some k, $0 \leq k \leq n-1$, where the theorem reduces to Theorem 6.1.

Corollary 6.7. *Let* $b_\lambda^\Lambda(q)$ *be the branching function of the level* l *integrable module* $\mathcal{L}(\Lambda)$, $\Lambda = \Lambda(K)$. *Then*

$$b_\lambda^\Lambda(q) = \lim_{m\to\infty} q^{-B_{K,m}} G_{\lambda_m/\nu_{K,m},(l^m)}(q),$$

where $B_{K,m} = \sum_{i=1}^m ik_i$ *and* $\lambda_m = \lambda + \left(\left(\frac{lm+|\nu_{K,m}|-|\lambda|}{n}\right)^n\right)$.

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